Recurrence Relations

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Problem 1

A)

$$2, 8, 32, 128, 512, \ldots$$

i) Give the recurrence relation for this sequence.

ii) Find a formula in terms of n for each term u_n.

B) A sequence is defined by the recurrence relation $u_n = ru_{n-1}$ and $u_0 = a$ Find the formula (involving n, r and a) for the nth term.

Problem 2

A The Fibonacci Sequence is defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$ $F_1 = 1$

i) Find the formula for F_n, the nth Fibonacci number (by using the auxiliary equation).

ii) By substituting the recurrence relation formula in place of the numerator of $\frac{F_n}{F_{n-1}}$, find the limit of the ratio $\frac{F_n}{F_{n-1}}$ as $n \to \infty$.

iii) If we are only given F_n, and n is large, what can we multiply this by to get F_{n+1}?

Try this by finding F_{11} given that $F_{10} = 55$.

Part A(iv) (as well as Problem 3) is left as an optional extra. If you're interested please read the explanation of continued fractions at the end of these problems, or alternatively ask me to explain them to you after the talk! Note: problem 3 does not need knowledge of continued fractions. It is a similar question to Problem 2 part B)

iv) Using your answer to part **A(ii)**, find the golden ratio, φ, as a continued fraction.

B) I can climb stairs one or two steps at a time. How many ways can I climb a flight of n stairs?

i) Start by counting the number of ways I can climb a flight of 1,2,3,4, and 5 stairs. Is there any recurrence relation between these you can notice?

Does this work generally? If so, why?

ii) By comparing this recurrence relation to the one in part **A(i)**, write down the formula for the number of ways I can climb a flight of n stairs.

Problem 3

A circle is divided into n sectors by drawing n radii. Show that the number of ways of using three colours so that neighbouring sectors are coloured differently is

$$2^n + 2(-1)^n$$

Explanation of continued fractions

A number, say $\frac{221}{41}$ can be written as a continued fraction by writing:

$$\frac{221}{41} = 5 + \frac{16}{41} = 5 + \frac{1}{\frac{41}{16}} = 5 + \frac{1}{2 + \frac{9}{16}} = 5 + \frac{1}{2 + \frac{1}{\frac{16}{9}}} = 5 + \frac{1}{2 + \frac{1}{1 + \frac{7}{9}}} = 5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{9}{7}}}}$$

$$= 5 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{7}}}} = 5 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{7}{2}}}}} = 5 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}}}$$

In general, continued fractions are normally written in the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}$$

However some continued fractions are never-ending such as in the following example:

$$\sqrt{2} = \frac{\sqrt{2}\left(1 + \sqrt{2}\right)}{1 + \sqrt{2}} = \frac{1 + \sqrt{2} + 2 - 1}{1 + \sqrt{2}} = 1 + \frac{2 - 1}{1 + \sqrt{2}}$$

It's clear that this process repeats if we do the same with the $\sqrt{2}$ on the denominator of the RHS and so

$$\sqrt{2} = 1 + \cfrac{2 - 1}{1 + 1 + \cfrac{2 - 1}{1 + \sqrt{2}}} = 1 + \cfrac{2 - 1}{2 + \cfrac{2 - 1}{1 + \sqrt{2}}}$$

We can repeat this process infinitely to give

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}$$

Incidentally, we can do this with any number, N:

$$\sqrt{N} = \frac{\sqrt{N}\left(1+\sqrt{N}\right)}{1+\sqrt{N}} = \frac{1+\sqrt{N}+N-1}{1+\sqrt{N}} = 1 + \frac{N-1}{1+\sqrt{N}}$$

It's clear that this process repeats if we do the same with the $\sqrt{N}$ on the denominator of the RHS and so

$$\sqrt{N} = 1 + \cfrac{N-1}{1 + 1 + \cfrac{N-1}{1+\sqrt{N}}} = 1 + \cfrac{N-1}{2 + \cfrac{N-1}{1+\sqrt{N}}}$$

We can repeat this process infinitely to give

$$\sqrt{N} = 1 + \cfrac{N-1}{2 + \cfrac{N-1}{2 + \cfrac{N-1}{2 + \cfrac{N-1}{2 + \cfrac{N-1}{\ddots}}}}}$$

Continued fractions have many uses including approximating irrational numbers (in the above example $\sqrt{2}$) by taking successive terms of the continued fraction. You may have noticed that if a number has a terminating continued fraction (i.e. not "never-ending"), then it is rational. Does the converse apply (i.e does a non-terminating continued fraction imply an irrational number)?